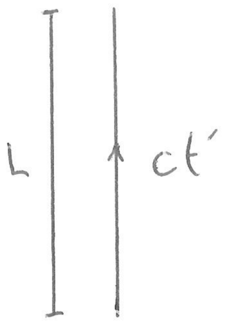
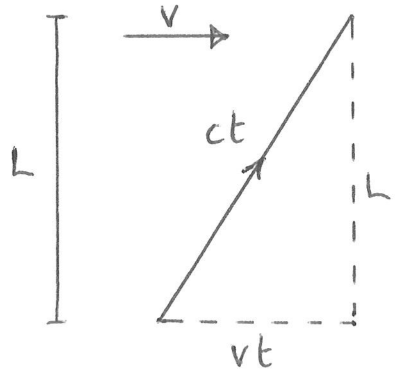
Time Dilation - The Lorentz Transformation



In a Stationary (rest) frame, light travels distance L in time t' as measured in the rest frame

Distance = speed × time so $L = ct'$



In a frame moving at velocity v with respect to the rest frame, light travels distance ct when observed from the rest frame.

c is invariant - it is constant when viewed from any inertial frame.

L and v are perpendicular so L is the same in both frames.

$$L = ct' \quad \text{and} \quad c^2t^2 = v^2t^2 + L^2$$

$$\text{so} \quad c^2t^2 = v^2t' + c^2t'^2$$

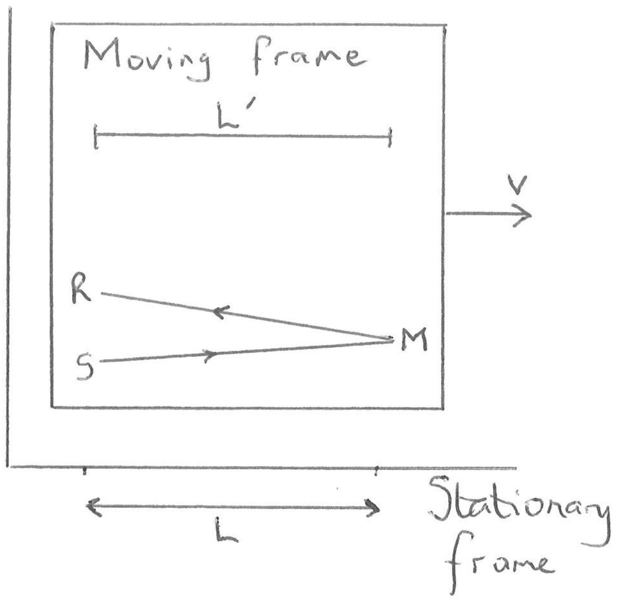
Gather t^2 and factorise: $t^2(c^2 - v^2) = c^2t'^2$

Divide by $(c^2 - v^2)$: $$t^2 = \frac{c^2t'^2}{c^2 - v^2}$$

Divide by $\frac{c^2}{c^2}$: $$t^2 = \frac{c^2t'^2/c^2}{(c^2 - v^2)/c^2} = \frac{t'^2}{1 - \frac{v^2}{c^2}}$$

Root both sides: $$t = \frac{t'}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Length Contraction



From the stationary frame, light appears to travel from the source S to the mirror M in time t and appears to travel a distance

$$L + vt_1$$

From the mirror to the receiver R it is

$$L - vt_2$$

Since time is invariant $L + vt_1 = ct_1$

so $t_1 = \dfrac{L}{c-v}$

and $L - vt_2 = ct_2$

so $t_2 = \dfrac{L}{c+v}$

The total time for the light to travel from S to R is $t_1 + t_2$

$$t = t_1 + t_2 = \frac{L}{c-v} + \frac{L}{c+v}$$

$$= \frac{L(c+v) + L(c-v)}{(c-v)(c+v)}$$

$$= \frac{2Lc}{c^2 - v^2}$$

Divide by $\frac{c^2}{c^2}$, $t = \dfrac{2Lc/c^2}{c^2 - v^2/c^2}$

so $t = \dfrac{2L/c}{1 - v^2/c^2}$ (1)

In the moving frame $2L' = ct'$

so $t' = \dfrac{2L'}{c}$

But from time dilation we know that

$$t' = t\sqrt{1 - v^2/c^2}$$

so $\dfrac{2L'}{c} = t\sqrt{1 - v^2/c^2}$

Subbing in (1) from above: $\dfrac{2L'}{c} = \dfrac{2L/c \, \sqrt{1 - v^2/c^2}}{1 - v^2/c^2}$

$\frac{2}{c}$ cancels: $L' = \dfrac{L}{\sqrt{1 - v^2/c^2}}$

$$\boxed{t = \frac{t'}{\gamma} \qquad L = \gamma L' \qquad \gamma = \sqrt{1 - v^2/c^2}}$$